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Casimir invariants for the complete family of quasisimple orthogonal algebras

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Abstract. A complete choice of generators of the centre of the enveloping algebras of real quasisimple Lie algebras of orthogonal type, for arbitrary dimension, is obtained in a unified setting. The results simultaneously include the well known polynomial invariants of the pseudo-orthogonal algebras so(p,q), as well as the Casimirs for many non-simple algebras such as the inhomogeneous iso(p,q), the Newton–Hooke and Galilei type, etc, which are obtained by contraction(s) starting from the simple algebras so(p,q). The dimension of the centre of the enveloping algebra of a quasisimple orthogonal algebra turns out to be the same as for the simple so(p,q) algebras from which they come by contraction. The structure of the higher-order invariants is given in a convenient 'pyramidal' manner, in terms of certain sets of 'Pauli–Lubanski' elements in the enveloping algebras. As an example showing this approach at work, the scheme is applied to recovering the Casimirs for the $(3+1)$-kinematical algebras. Some prospects on the relevance of these results for the study of expansions are also given.

1. Introduction

The role of Casimir (or polynomial) invariants of Lie algebras is rather important in physics as well as in mathematics. They generate the centre of the universal enveloping algebra Ug of g. Physically, in any theory with a symmetry algebra, they appear as being related to conserved quantities, as they commute with all generators. The work of Racah [1] solved the problem of obtaining the Casimir invariants associated to a simple Lie algebra and, in particular, Gel'fand [2] explicitly found a particular basis of the centre of the enveloping algebra of so$(N+1)$; the case for the pseudo-orthogonal Lie algebras so(p,q) (in fact, for all simple classical algebras) has also been dealt with in an explicit form by Perelomov and Popov [3]. The number of independent Casimirs for simple Lie algebras is equal to the rank of the algebra; these Casimirs can be chosen as homogeneous polynomials in the generators, of degrees d_i which are related to the so-called exponents of the simple compact real form of the Lie algebra. For non-simple algebras, some general results have been established; for instance, a formula giving the number of primitive independent Casimir operators of any Lie algebra can be found in [4], and a complete description of the theory of polynomial and/or rational invariants appears in [5]. These results allow us to deduce all invariants related to any particular Lie algebra on a case-by-case basis, but do not give directly a general perspective of the structure of the invariants associated to a complete family of 'neighbour' algebras, as for instance provided by a given Lie algebra and (some of) its contractions: the general problem of relating the universal enveloping algebra of a given Lie algebra and one of its possible contractions is not yet solved in full generality,

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although complete results are available for special cases. For instance, Casimirs for a large family of contractions of $\mathrm{sl}(3, \mathbb{C})$ are given in [6], the question of graded contractions of Casimir operators is addressed within a general framework in [7] and the behaviour under contraction of bilinear invariant forms, which is directly related to the quadratic Casimirs, is studied within the graded contraction approach in [8]. This problem has definite interest in physics, where contractions are related to some kind of 'approximation' and understanding how invariants behave under contraction and under the 'inverse' expansion or deformation process are illuminating aspects of the theory.

The aim of this paper is to obtain, within such a 'simultaneous' approach, all the Casimir invariants for every Lie algebra in a rather large family, the so-called quasisimple or Cayley–Klein (CK) algebras of orthogonal type [9, 10]. This family includes the real simple pseudo-orthogonal Lie algebras $\mathrm{so}(p, q)$ of the Cartan series B_l and D_l as well as many non-simple Lie algebras which can be obtained by contracting the former ones. The complete family of quasisimple orthogonal algebras can be obtained starting from the compact algebra $\mathrm{so}(N + 1)$ in two different ways. One possibility is to use a 'formal transformation' which introduces numbers outside the real field (either complex, double or dual (Study) numbers) [11], and another uses the theory of graded contractions [12, 13], without leaving the real field. Adopting this last point of view, it is shown that a particular solution of the $\mathbb{Z}_2^{\otimes N}$ graded contractions of $\mathrm{so}(N + 1)$ leads to the CK algebras as an N-parametric family of real Lie algebras denoted $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N + 1)$ [14]. The Lie algebra structure of the above family together with a listing of its most interesting members are briefly described in section 2.

When all ω_a are different from zero, $so_{\omega_1,\ldots,\omega_N}(N+1)$ is a simple algebra (isomorphic to $\mathrm{so}(p, q)$ with $p+q = N+1$), whose rank is $l = [\frac{N+1}{2}]$ (the square brackets here denoting, as usual, the integer part). The dimension of the centre of its universal enveloping algebra equals the rank l of the algebra, and it is generated by a set of homogeneous polynomials (Casimir operators) of orders $2, 4, \ldots, 2[\frac{N}{2}]$, and an additional Casimir of order l when $N+1$ is even. In section 3 we present the explicit structure of the Casimir invariants corresponding to any algebra in the family $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$. These invariants are deduced starting from the original approach of Gel'fand but where the necessary modifications are introduced in order to get expressions which cover simultaneously all algebras in the family $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N + 1)$, whether the constants ω_a are different from zero or not. This means that the behaviour of these Casimirs upon any contraction $\omega_a \to 0$ is built-in in the formalism, and they do not require any rescaling which should be made when the contraction is performed in the Inönü–Wigner sense. Every Casimir we obtain is non-trivial for any contracted algebra (whether or not the constants ω_a are different from zero); furthermore, these constitute a complete set of Casimirs for CK algebras.

The main tool is provided by some elements in the enveloping algebra, labelled by an even number $2s$ of indices, $W_{a_1a_2\ldots a_s b_1 b_2\ldots b_s}$, which are homogeneous of order s in the generators. In the case of the (3+1) Poincaré algebra, the components of the Pauli–Lubanski vector (whose square is the fourth-order Casimir) are in fact W-symbols with four indices. In this way, the Casimir invariants are presented in a pyramidal intrinsic form since each $W_{a_1a_2\ldots a_s a_{s+1} b_1 b_2\ldots b_s b_{s+1}}$ can be written in terms of W's with two less indices $W_{a_1a_2\ldots a_s b_1 b_2\ldots b_s}$, and ultimately, in terms of W-symbols with two indices, which are simply the generators themselves.

The problem of giving explicit expressions in terms of generators for Casimirs in CK algebras has also been approached by Gromov [15] by applying the above-mentioned formal transformation to the Casimir invariants of $\mathrm{so}(N + 1)$ obtained by Gel'fand: those expressions should be equivalent to those we shall obtain (as giving a possibly different basis for the centre of the enveloping algebra), but the explicit introduction of the W's makes the choice in this paper a lot simpler and easily tractable. The general expressions for Casimirs written directly in terms of generators, as in [15], are overall more cumbersome to apply to specific Lie algebras than those involving W's, especially when N increases.

In section 4 the results are illustrated by writing the general expressions of the invariants associated to $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ for $N=2,3,4,5$; in particular, for $N=4$ we focus on the (3+1)-kinematical algebras [16] thus obtaining a global view of the limit transitions among their corresponding Casimir operators. The way of getting the Casimirs in the Minkowski space starting from those in the de Sitter space by letting the universe 'radius' $R\to\infty$ is well known; this familiar example appears in our scheme as a rather particular case, yet it may facilitate grasping the scope of the results we obtain, which includes a much larger family of algebras than the well known kinematical ones.

In the conclusions (section 5) we make some brief comments on the role of these results for the study of expansions.

2. The family of quasisimple orthogonal algebras

Consider the real Lie algebra $\mathrm{so}(N+1)$ whose $\frac{1}{2}N(N+1)$ generators Ω_{ab} $(a,b=0,1,\ldots,N, a<b)$ satisfy the following non-vanishing Lie brackets:

$$[\Omega_{ab},\Omega_{ac}]=\Omega_{bc}\qquad[\Omega_{ab},\Omega_{bc}]=-\Omega_{ac}\qquad[\Omega_{ac},\Omega_{bc}]=\Omega_{ab}\qquad a<b<c. \tag{2.1}$$

Through a $\mathbb{Z}_2^{\otimes N}$ graded contraction process, a family of contracted real Lie algebras can be deduced from $\mathrm{so}(N+1)$. The general solution was given in [17]; it includes a range of algebras, from the simple Lie algebras $\mathrm{so}(p,q)$ to the Abelian algebra of the same dimension. For reasons which will become clear shortly, we restrict ourselves here to a particular subfamily [14], whose members have been called quasisimple algebras [9] because they are very 'near' the simple ones. They depend on N real coefficients $\omega_1,\ldots,\omega_N$ and the generic member of this family will be denoted $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$. Their non-zero commutators are given by

$$[\Omega_{ab},\Omega_{ac}]=\omega_{ab}\Omega_{bc}\qquad[\Omega_{ab},\Omega_{bc}]=-\Omega_{ac}\qquad[\Omega_{ac},\Omega_{bc}]=\omega_{bc}\Omega_{ab}\qquad a<b<c \tag{2.2}$$

without summing over repeated indices. Note that all Lie brackets involving four different indices a, b, c, d as $[\Omega_{ab},\Omega_{cd}]$ are equal to zero.

The two-index coefficients ω_{ab} are written in terms of the N basic ω_a by means of:

$$\omega_{ab}=\omega_{a+1}\omega_{a+2}\cdots\omega_b\qquad a,b=0,1,\ldots,N\qquad a<b \tag{2.3}$$

therefore

$$\omega_{a-1\,a}=\omega_a\qquad a=1,\ldots,N \tag{2.4}$$

$$\omega_{ac}=\omega_{ab}\omega_{bc}\qquad a<b<c. \tag{2.5}$$

Each coefficient ω_a can be reduced to $+1$, -1 or 0 by simply rescaling the initial generators; then the family $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ embraces 3^N Lie algebras called CK algebras of orthogonal type or quasisimple orthogonal algebras. Some of them can be isomorphic; a useful isomorphism is:

$$\mathrm{so}_{\omega_1,\omega_2,\ldots,\omega_{N-1},\omega_N}(N+1)\simeq\mathrm{so}_{\omega_N,\omega_{N-1},\ldots,\omega_2,\omega_1}(N+1). \tag{2.6}$$

The algebra $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ has a (vector) representation by $(N+1)\times(N+1)$ real matrices, given by

$$\Omega_{ab}=-\omega_{ab}e_{ab}+e_{ba} \tag{2.7}$$

where e_{ab} is the matrix with a single non-zero entry, 1, in the row a and column b. In this realization, any element $X \in so_{\omega_1,\ldots,\omega_N}(N+1)$ satisfies the equation:

$$X I_\omega + I_\omega{}^t X = 0 \tag{2.8}$$

where I_ω is the diagonal matrix

$$I_\omega = \mathrm{diag}(+, \omega_{01}, \omega_{02}, \ldots, \omega_{0N}) = \mathrm{diag}(+, \omega_1, \omega_1\omega_2, \ldots, \omega_1 \cdots \omega_N) \tag{2.9}$$

and ${}^t X$ means the transpose matrix. We state this property by saying that X is an I_ω-antisymmetric matrix; when all $\omega_a = 1$, this reduces to the standard antisymmetry for the generators of $\mathrm{so}(N+1)$.

The CK algebras $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ are the Lie algebras of the motion groups of N-dimensional symmetrical homogeneous spaces $\mathcal{X}_0$:

$$\mathcal{X}_0 \equiv \mathrm{SO}_{\omega_1,\ldots,\omega_N}(N+1)/\mathrm{SO}_{\omega_2,\ldots,\omega_N}(N) \tag{2.10}$$

where the subgroup $H_0 \equiv \mathrm{SO}_{\omega_2,\ldots,\omega_N}(N)$ is generated by the Lie subalgebra $h_0 = \langle \Omega_{ab}, a, b = 1, \ldots, N \rangle$. Each space $\mathcal{X}_0$ has constant curvature equal to ω_1 and its principal metric can be reduced to the form $\mathrm{diag}(+, \omega_2, \omega_2\omega_3, \ldots, \omega_2 \cdots \omega_N)$ at each point.

In the sequel we identify the most interesting Lie algebras appearing within $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ according to the cancellation of some ω_a [18, 19]. In particular, the kinematical algebras [16] associated to different models of spacetime are CK algebras. In the list below, when we explicitly say that if some coefficient is equal to zero it will be understood that the remaining ones are not. It is remarkable that each case $\omega_a = 0$ can be regarded as an Inönü–Wigner contraction limit, where some parameter $\varepsilon_a \to 0$ [14, 20].

(1) $\omega_a \neq 0\ \forall a$. They are the pseudo-orthogonal algebras $\mathrm{so}(p, q)$ with $p + q = N + 1$ of the Cartan series B_l or D_l. The quadratic form invariant under the fundamental vector representation is given by the matrix I_ω (2.9).

(2) $\omega_1 = 0$. They are inhomogeneous pseudo-orthogonal algebras $\mathrm{iso}(p, q)$ with $p + q = N$ which have a semidirect sum structure:

$$\mathrm{so}_{0,\omega_2,\ldots,\omega_N}(N+1) \equiv t_N \odot \mathrm{so}_{\omega_2,\ldots,\omega_N}(N) \equiv \mathrm{iso}(p, q).$$

The 'signature' of the metric invariant under $\mathrm{so}_{\omega_2,\ldots,\omega_N}(N)$ is $(+, \omega_{12}, \omega_{13}, \ldots, \omega_{1N})$. The most interesting cases are the Euclidean algebra $\mathrm{iso}(N)$ which is recovered once for $(\omega_1, \omega_2, \ldots, \omega_N) = (0, +, \cdots, +)$, and the Poincaré algebra $\mathrm{iso}(N-1, 1)$ which appears several times, for instance, for $(\omega_1, \omega_2, \ldots, \omega_N) = \{(0, -, +, \cdots, +),\ (0, +, \cdots, +, -),\ (0, +, \cdots, +, -, -, +, \cdots, +),\ (0, -, -, +, \cdots, +),\ (0, +, \cdots, +, -, -)\}$. From the isomorphism (2.6) is clear that the CK algebras with $\omega_N = 0$ are also of this kind and similarly for the next types.

(3) $\omega_1 = \omega_2 = 0$. They are twice inhomogeneous pseudo-orthogonal algebras $\mathrm{iiso}(p, q)$ with $p + q = N - 1$:

$$\mathrm{so}_{0,0,\omega_3,\ldots,\omega_N}(N+1) \equiv t_N \odot (t_{N-1} \odot \mathrm{so}_{\omega_3,\ldots,\omega_N}(N-1)) \equiv \mathrm{iiso}(p, q).$$

The signature of $\mathrm{so}_{\omega_3,\ldots,\omega_N}(N-1)$ is $(+, \omega_{23}, \omega_{24}, \ldots, \omega_{2N})$. Hence the Galilean algebra $\mathrm{iiso}(N-1)$ is associated to $(0, 0, +, \cdots, +)$.

(4) $\omega_1 = \omega_N = 0$. They are $\mathrm{ii'so}(p, q)$ algebras with $p + q = N - 1$:

$$\mathrm{so}_{0,\omega_2,\ldots,\omega_{N-1},0}(N+1) \equiv t_N \odot (t'_{N-1} \odot \mathrm{so}_{\omega_2,\ldots,\omega_{N-1}}(N-1)) \equiv \mathrm{ii'so}(p, q)$$

where $\mathrm{so}(p, q)$ acts on t_N through the vector representation while it acts on t'_{N-1} through the contragredient of the vector representation. Thus, $\mathrm{ii'so}(N-1)$ is the Carroll algebra with coefficients $(0, +, \cdots, +, 0)$ [16].

(5) $\omega_a = 0$, $a \neq 1, N$. They are the $t_r(\mathrm{so}(p,q)\oplus\mathrm{so}(p',q'))$ algebras [21]. In particular, for $\omega_2 = 0$ we have $t_{2N-2}(\mathrm{so}(p,q)\oplus\mathrm{so}(p',q'))$ with $p+q = N-1$ and $p'+q' = 2$, which include the expanding and oscillating Newton–Hooke algebras for $q = 0$ [16].

(6) When all coefficients $\omega_a = 0$ we find the flag space algebra $\mathrm{so}_{0,\ldots,0}(N+1) \equiv \mathrm{i}\ldots\mathrm{iso}(1)$ [9].

3. The Casimir invariants

We first recall the necessary definitions and tools [4, 5]. Then we summarize the approach and the results of Gel'fand [2] for $\mathrm{so}(N+1)$. Afterwards we compute the Casimir invariants for the CK algebras $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$.

3.1. Definition of polynomial invariants

In this paragraph, g will denote any Lie algebra, of dimension D. Let Ug be the *enveloping algebra* of g generated by all polynomials in the generators X_μ $\mu = 1,\ldots,D$ and S the *symmetric algebra* of g isomorphic to $\mathbb{R}[\alpha_\mu;\ \mu = 1,\ldots,D]$, this is, the ring of polynomials in D commutative variables α_μ. A generic polynomial is denoted as $p = p(\alpha_1,\ldots,\alpha_D)$.

The *adjoint action*, $\mathrm{ad}\, X_\mu : g \longrightarrow g$

$$\mathrm{ad}\, X_\mu(X_\nu) = [X_\mu, X_\nu] \tag{3.1}$$

is extended to an 'adjoint action' of g on Ug and also to another action on S:

$$\mathrm{ad}\, X_\mu : u \in Ug \longrightarrow [X_\mu, u] \equiv X_\mu u - u X_\mu \in Ug \tag{3.2}$$

$$\mathrm{ad}\, X_\mu : p = p(\alpha_1,\ldots,\alpha_D) \in S \longrightarrow \mathcal{O}_\mu(p) \equiv \sum_{\nu,\sigma=1}^{D} C^\sigma_{\mu,\nu}\alpha_\sigma \frac{\partial p}{\partial \alpha_\nu} \in S \tag{3.3}$$

where $C^\sigma_{\mu,\nu}$ are the structure constants of g.

The invariants in Ug and S under the adjoint action of g are the following subsets:

$$U^I g \equiv \{u \in Ug | [X_\mu, u] = 0, \forall X_\mu \in g\} \subset Ug \tag{3.4}$$

$$S^I \equiv \{p \in S | \mathcal{O}_\mu(p) = 0, \forall X_\mu \in g\} \subset S \tag{3.5}$$

and the elements of $U^I g$ are called *polynomial* or *Casimir invariants* of g. According to the general results described in [4, 5] the two main steps to obtain the Casimir invariants are:

- to compute the subset S^I of S;
- to apply to each element of S^I the canonical mapping $\phi : S \to Ug$ defined for any monomial by symmetrization:

$$\phi(\alpha_{\mu_1}\ldots\alpha_{\mu_r}) = \frac{1}{r!}\sum_{\pi\in\Pi_r} X_{\pi(\mu_1)}\cdots X_{\pi(\mu_r)} \tag{3.6}$$

where Π_r is the group of permutations on r items, and extended to S by linearity.

A *general* result providing an upper bound for the number of independent invariants for g is as follows.

Proposition 1. [5]. The maximal number of algebraically independent Casimir invariants τ associated to any Lie algebra g is

$$\tau \leqslant \dim(g) - r(g) \tag{3.7}$$

where $\dim(g)$ is the dimension of g, and $r(g)$ is the rank of the antisymmetric matrix M_g whose elements are

$$(M_g)_{\mu,\nu} = \sum_{\sigma=1}^{D} C^{\sigma}_{\mu,\nu}\alpha_\sigma. \tag{3.8}$$

For the simple Lie algebras $\mathrm{so}(p,q)$, $p+q = N+1$, the number of algebraically independent Casimir invariants is equal to $[\frac{N+1}{2}]$, the rank of the algebra. In this case, (3.7) holds as an equality, $\tau = \dim(g) - r(g)$, and this can be checked by computing the rank $r(g)$ of matrix (3.8) (see below). However, it is not necessary that g is simple in order to have the equality $\tau = \dim(g) - r(g)$. For instance, the Poincaré algebra $\mathrm{iso}(3,1)$ has two algebraically independent Casimirs (quadratic and fourth order), and in this case the equality also holds; the same happens for all the inhomogenous pseudo-orthogonal algebras $\mathrm{iso}(p,q)$ which appear in the CK family when $\omega_1 = 0$ or $\omega_N = 0$ and the remaining ones are different from zero. In the extreme contracted case of the Abelian algebra, $r(g) = 0$ and $\tau = \dim(g)$ saturates inequality (3.7) again.

3.2. The Gel'fand method for so(N + 1)

Now we restrict our attention to our the compact real Lie algebra $\mathrm{so}(N+1)$, as a distinguished member of the CK family of algebras, whose generators will be denoted as in (2.1). Hereafter, when referring to equations in section 3.1, the obvious changes $X_\mu \to \Omega_{ab}$, $\alpha_\mu \to \alpha_{ab}$ should be understood. Instead of solving the set of differential equations (3.5) in order to get the elements of S^I, Gel'fand considered the following antisymmetric matrix associated to $\mathrm{so}(N+1)$:

$$T = \begin{pmatrix} 0 & \alpha_{10} & \alpha_{20} & \ldots & \alpha_{N-10} & \alpha_{N0} \\ \alpha_{01} & 0 & \alpha_{21} & \ldots & \alpha_{N-11} & \alpha_{N1} \\ \alpha_{02} & \alpha_{12} & 0 & \ldots & \alpha_{N-12} & \alpha_{N2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{0N-1} & \alpha_{1N-1} & \alpha_{2N-1} & \ldots & 0 & \alpha_{NN-1} \\ \alpha_{0N} & \alpha_{1N} & \alpha_{2N} & \ldots & \alpha_{N-1N} & 0 \end{pmatrix} \tag{3.9}$$

where $\alpha_{ab} = -\alpha_{ba}$. He obtained the Casimir invariants of $\mathrm{so}(N+1)$ from the coefficients of the characteristic polynomial of the matrix T:

$$\det(T - \lambda I) = 0 \tag{3.10}$$

where I is the $(N+1)\times(N+1)$ identity matrix. Due to the structure of the matrix T, these coefficients are sums of all minors of the same order associated to the main diagonal of T; the last coefficient is of course the determinant of T [2]. It turns out that this determinant is equal to zero when N is even $N = 2l$, and it is a perfect square of an homogeneous expression of order l in the variables α_{ab} when N is odd $N = 2l-1$. The Casimirs themselves are obtained through the replacement $\alpha_{ab} \to \Omega_{ab}$ and further symmetrization on the variables α_{ab} in these coefficients.

When $N = 2l$ is even, Gel'fand obtained l such invariants, $\mathcal{C}_1, \ldots, \mathcal{C}_s, \ldots, \mathcal{C}_l$ for $\mathrm{so}(N+1)$ which are homogeneous polynomials of order $2s$ in the generators:

$$\mathcal{C}_s = \sum_{i_1,i_2,\ldots,i_{2s-1},i_{2s}=0}^{N} \Omega_{i_1 i_2}\Omega_{i_2 i_3}\ldots\Omega_{i_{2s-1}i_{2s}}\Omega_{i_{2s}i_1} \qquad s = 1,2,\ldots,l. \tag{3.11}$$

When N is odd, $N = 2l - 1$, besides (3.11) there is another invariant coming from the determinant of T, which is a perfect square of an homogeneous expression in the generators of order l denoted simply as $\mathcal{C}$:

$$\mathcal{C} = \sum_{i_0, i_1, \ldots, i_N = 0}^{N} \varepsilon_{i_0 i_1 \ldots i_N} \Omega_{i_0 i_1} \Omega_{i_2 i_3} \ldots \Omega_{i_{N-1} i_N} \qquad (3.12)$$

where $\varepsilon_{i_0 i_1 \ldots i_N}$ is the completely antisymmetric unit tensor. In both expressions the relation $\Omega_{ab} = -\Omega_{ba}$ is understood.

The degrees d_i of these Casimirs follow also from their relation with the exponents of $SO(N+1)$; for this connection see [22]. The Poincaré polynomial $\sum_i b_i t^i$ of the compact real form of any simple Lie group space has the form $\prod_i (1 + t^{a_i})$, and the exponents a_i appearing in this product are related to the degrees d_i of the Casimirs of the Lie algebra of the group by $a_i = 2d_i - 1$. It is thus possible to foresee the degrees of Casimirs of simple Lie algebras, provided the exponents are known. These follow from the knowledge of real homology of the group spaces, and since Pontrjagin and Hopf the mathematicians have known that as far as real homology is concerned, compact forms of simple Lie groups behave like products of odd-dimensional spheres, whose dimensions are the exponents a_i. For each such sphere S^{a_i}, there is an a_i-skew form in the Lie algebra, which is related to a $(a_i + 1)/2$-multilinear symmetric form, the link being provided by the construction of Chevalley and Weil (see, e.g. [23]). And finally, each such multilinear form is dual to a Casimir of the same degree.

3.3. The case of orthogonal CK algebras

We now proceed to implement this scheme for the CK algebras. Before going into detail, let us first comment upon the results. For *any* CK algebra $so_{\omega_1,\ldots,\omega_N}(N+1)$ in the CK family, the maximal number of algebraically independent Casimirs (i.e. the dimension of the centre of the universal enveloping algebra of $so_{\omega_1,\ldots,\omega_N}(N+1)$ on its own), is still given by the same value $[\frac{N+1}{2}]$ as in the simple case, no matter how many constants ω_i are equal to zero or not. This property of having exactly $[\frac{N+1}{2}]$ algebraically independent Casimirs justifies the *quasisimple* name allocated to the CK algebras: although the CK set contains non-simple algebras, all members in each family have the same number of algebraically independent Casimirs. This is the main reason of restricting ourselves, in this paper, from the general set of graded contractions of $so(N+1)$ to the subfamily of CK algebras. This property is no longer true for other contractions of $so(N+1)$ beyond the CK family, and the dimension of the centre of the universal enveloping algebra of such contracted algebras is in general larger than $[\frac{N+1}{2}]$, as the extreme case of the Abelian algebra, with as many primitive Casimirs as generators, clearly shows.

As a first step we write the analogous of T (3.9) *only* for the pseudo-orthogonal $so(p,q)$ algebras *with all* $\omega_a \neq 0$ (but *without* reducing them to ± 1), and compute the minors separately. Afterwards, we arrange the details, by introducing some factors depending on the coefficients ω_a in such way that all contractions $\omega_a \to 0$ are always well defined and do not originate a trivial result for any of the Casimirs found.

Consider first the CK algebras with all $\omega_a \neq 0$. Recall that the generators of $so_{\omega_1,\ldots,\omega_N}(N+1)$ have been taken as Ω_{ab} only for $a < b$. If all $\omega_a \neq 0$ we can extend this set and introduce the (linearly dependent) new generators by defining Ω_{ba} with $a < b$, and their corresponding α_{ba} as follows

$$\text{if } a < b \qquad \Omega_{ba} := -\frac{1}{\omega_{ab}} \Omega_{ab} \qquad \alpha_{ba} := -\frac{1}{\omega_{ab}} \alpha_{ab} \qquad (3.13)$$

so that the commutation relations (2.2) can be written in the standard form

$$[\Omega_{ab}, \Omega_{lm}] = \delta_{am}\Omega_{lb} - \delta_{bl}\Omega_{am} + \delta_{bm}\omega_{lm}\Omega_{al} + \delta_{al}\omega_{ab}\Omega_{bm} \tag{3.14}$$

which is the familiar form of the commutation relations for an $\mathrm{so}(p, q)$ algebra, with the non-degenerate metric tensor (2.9). In this special case $\omega_a \neq 0$ we associate to $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ the matrix (3.9) denoted $T_{\omega_1,\ldots,\omega_N}$ which now reads

$$T_{\omega_1,\ldots,\omega_N} = \begin{pmatrix} 0 & -\frac{\alpha_{01}}{\omega_{01}} & -\frac{\alpha_{02}}{\omega_{02}} & \ldots & -\frac{\alpha_{0N-1}}{\omega_{0N-1}} & -\frac{\alpha_{0N}}{\omega_{0N}} \\ \alpha_{01} & 0 & -\frac{\alpha_{12}}{\omega_{12}} & \ldots & -\frac{\alpha_{1N-1}}{\omega_{1N-1}} & -\frac{\alpha_{1N}}{\omega_{1N}} \\ \alpha_{02} & \alpha_{12} & 0 & \ldots & -\frac{\alpha_{2N-1}}{\omega_{2N-1}} & -\frac{\alpha_{2N}}{\omega_{2N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{0N-1} & \alpha_{1N-1} & \alpha_{2N-1} & \ldots & 0 & -\frac{\alpha_{N-1N}}{\omega_{N-1N}} \\ \alpha_{0N} & \alpha_{1N} & \alpha_{2N} & \ldots & \alpha_{N-1N} & 0 \end{pmatrix}. \tag{3.15}$$

This matrix satisfies the property

$$T_{\omega_1,\ldots,\omega_N} I_\omega + I_\omega{}^t T_{\omega_1,\ldots,\omega_N} = 0 \tag{3.16}$$

where I_ω is the diagonal matrix (2.9). Recalling that (2.8), we say that $T_{\omega_1,\ldots,\omega_N}$ is an I_ω-antisymmetric matrix.

When *all* the constants ω_a are different from zero, it is clear that the Casimir invariants for $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ are the coefficients of the characteristic polynomial coming from equation (3.10) where T is replaced by $T_{\omega_1,\ldots,\omega_N}$. In order to get them we have to calculate the determinant of a generic diagonal submatrix, which will have the same structure as (3.15) but with a non-consecutive subset of indices, say $T_{\omega_{i_1 i_2},\ldots,\omega_{i_{K-1} i_K}}$. Due to property (3.16) it is easy to show that for an odd K the determinant is always zero. Therefore, only determinants for even $K = 2s$ might be different from zero. For future convenience, we will denote any arrangement of $2s$ indices taken from $012\ldots N$ in increasing order as $a_1 < a_2 < \cdots < a_s < b_1 < b_2 < \cdots < b_s$.

We now define some symbols of $2s$ indices $\mathcal{W}_{a_1 a_2 \ldots a_s b_1 b_2 \ldots b_s}$, for $s = 1, 2, \ldots, l$ as:

$$\mathcal{W}^2_{a_1 a_2 \ldots a_s b_1 b_2 \ldots b_s} := \omega_{a_1 b_s}\omega_{a_2 b_{s-1}} \ldots \omega_{a_s b_1} \det[T_{\omega_{a_1 a_2},\ldots,\omega_{a_s b_1},\ldots,\omega_{b_{s-1} b_s}}]. \tag{3.17}$$

This definition is justified since the r.h.s. of (3.17) is a perfect square. The set of coefficients ω_{ab} multiplying the determinant assures that the final expressions we are going to obtain are non-trivial even after the limits $\omega_a \to 0$. Inserting these ω factors turns out to be equivalent to the usual rescaling made in the contraction of Casimir invariants by means of an Inönü–Wigner contraction.

The $2s$-index $\mathcal{W}$-symbol is given in terms of the $(2s-2)$-index $\mathcal{W}$-symbol through

$$\begin{aligned}\mathcal{W}_{a_1 a_2 \ldots a_s b_1 b_2 \ldots b_s} = & \sum_{\mu=1}^{s} (-1)^{\mu+1} \alpha_{a_\mu b_s} \mathcal{W}_{a_1 a_2 \ldots \widehat{a_\mu} \ldots a_s b_1 b_2 \ldots \widehat{b_s}} \\ & + \sum_{\nu=1}^{s-1} (-1)^{s+\nu+1} \omega_{a_s b_\nu} \alpha_{b_\nu b_s} \mathcal{W}_{a_1 a_2 \ldots a_s b_1 b_2 \ldots \widehat{b_\nu} \ldots \widehat{b_s}}\end{aligned} \tag{3.18}$$

where the $\mathcal{W}$-symbols on the r.h.s. of the equation have $2s-2$ indices, those obtained by *removing* the two indices marked with a caret a_μ, b_s or b_ν, b_s from the set of $2s$ indices $a_1 a_2 \ldots a_s b_1 b_2 \ldots b_s$.

The $\mathcal{W}$-symbols give rise to the elements of S^I (3.5) and the canonical mapping ϕ (3.6) transforms them into invariants of the enveloping CK algebra (3.4). The symmetrization implied by action of ϕ on $\mathcal{W}$ reduces to a simple substitution $\alpha_{ab} \to \Omega_{ab}$ since all generators appearing in the products of the $\mathcal{W}$-symbols commute. Once the substitution of the variables α_{ab} by the generators Ω_{ab} has been performed, we will denote $W := \phi(\mathcal{W})$. Now $W_{ab}, W_{a_1a_2b_1b_2}, \ldots,$ are elements in the universal enveloping algebra of the CK Lie algebra. Their structure can be most clearly presented in a recursive way. For $a < b$, let:

$$W_{ab} := \Omega_{ab} \tag{3.19}$$

then W-symbols with four indices $a_1 < a_2 < b_1 < b_2$ are given in terms of those with two by:

$$W_{a_1a_2b_1b_2} = \Omega_{a_1b_2} W_{a_2b_1} - \Omega_{a_2b_2} W_{a_1b_1} + \omega_{a_2b_1} \Omega_{b_1b_2} W_{a_1a_2} \tag{3.20}$$

and further W-symbols with six, eight, ..., $2s$ indices, $W_{a_1a_2\ldots a_sb_1b_2\ldots b_s}$ (with $a_1 < a_2 < \cdots < a_s < b_1 < b_2 < \cdots < b_s$), are given in terms of those with two less indices by means of the relations:

$$\begin{aligned} W_{a_1a_2\ldots a_sb_1b_2\ldots b_s} = {} & \sum_{\mu=1}^{s} (-1)^{\mu+1} \Omega_{a_\mu b_s} W_{a_1a_2\ldots\widehat{a_\mu}\ldots a_sb_1b_2\ldots\widehat{b_s}} \\ & + \sum_{\nu=1}^{s-1} (-1)^{s+\nu+1} \omega_{a_sb_\nu} \Omega_{b_\nu b_s} W_{a_1a_2\ldots a_sb_1b_2\ldots\widehat{b_\nu}\ldots\widehat{b_s}} \end{aligned} \tag{3.21}$$

until we end up with W-symbols with $2l$ indices.

By using these W's we can produce expressions for the Casimir invariants of the CK algebras $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$. The key to this adaptation is to profit from the presence of the constants ω_a. When contraction is dealt with through an Inönü–Wigner-type contraction, a suitable rescaling of the non-contracted Casimir by some power of the contraction parameter is required to give a non-trivial well defined Casimir for the contracted algebra after the contraction limit. This is made unnecessary in our approach, which has this rescaling automatically built-in.

We now give the expressions, in terms of these W's, for the $[\frac{N+1}{2}]$ Casimir operators in the general CK Lie algebra $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$.

Theorem 2. The $l = [\frac{N+1}{2}]$ independent polynomial Casimir operators of the CK Lie algebra $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ can be written as:

- $[\frac{N}{2}]$ invariants $\mathcal{C}_s$, $s = 1, \ldots, [\frac{N}{2}]$ of order $2s$. We give the first, second, and then the general expression:

$$\mathcal{C}_1 = \sum_{\substack{a_1,b_1=0\\ a_1<b_1}}^{N} \omega_{0a_1}\omega_{b_1N} W_{a_1b_1}^2 \tag{3.22}$$

$$\mathcal{C}_2 = \sum_{\substack{a_1,a_2,b_1,b_2=0\\ a_1<a_2<b_1<b_2}}^{N} \omega_{0a_1}\omega_{1a_2}\omega_{b_1(N-1)}\omega_{b_2N} W_{a_1a_2b_1b_2}^2 \tag{3.23}$$

$$\mathcal{C}_s = \sum_{\substack{a_1,a_2,\ldots,a_s,b_1,b_2,\ldots,b_s=0\\ a_1<a_2<\cdots<a_s<b_1<b_2<\cdots<b_s}}^{N} \omega_{0a_1}\omega_{1a_2}\ldots\omega_{(s-1)a_s}\omega_{b_1(N-s+1)}\omega_{b_2(N-s+2)}\ldots\omega_{b_sN} W_{a_1a_2\ldots a_sb_1b_2\ldots b_s}^2. \tag{3.24}$$

- When $N+1$ is even, there is an extra Casimir $\mathcal{C}$ of order $l = \frac{N+1}{2}$:

$$\mathcal{C} = W_{012\ldots N}. \tag{3.25}$$

In these expressions any ω_{aa} should be understood as $\omega_{aa} := 1$. It is easy to see that even in the most contracted CK algebra, the flag space algebra, $\mathrm{so}_{0,\ldots,0}(N+1)$, these Casimirs are not trivial. In fact, the term in (3.24) with the W-symbol whose first group of s indices are consecutive and start from 0, and whose last group of s indices are also consecutive and end with N, $W^2_{012\ldots(s-1)(N-s+1)\ldots(N-2)(N-1)N}$, is the only term whose ω factor is equal to 1, and therefore the only one which survives in the Casimir $\mathcal{C}_s$ for the $\mathrm{so}_{0,\ldots,0}(N+1)$ algebra.

Therefore, theorem 2 provides a set of $[\frac{N+1}{2}]$ non-trivial independent Casimirs for any Lie algebra in the CK family. The question now is whether there exists any other Casimir which cannot be obtained by this contraction process. To answer this we should analyse the upper bound τ (3.7).

Proposition 3. The rank of the matrix defined by (3.8) for $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ is $N^2/2$ for even N and $(N^2-1)/2$ for odd N, regardless of the specific values of the coefficients ω_a.

Proof. We first read off from (2.2) the structure constants of the generic CK algebra:

$$C^{mn}_{ab,ac} = \delta_{mb}\delta_{nc}\omega_{ab} \qquad C^{mn}_{ab,bc} = -\delta_{ma}\delta_{nc} \qquad C^{mn}_{ac,bc} = \delta_{ma}\delta_{nb}\omega_{bc} \tag{3.26}$$

where the conditions $a < b < c$ and $m < n$ will be assumed without saying. The result stated in proposition 3 follows from the existence of structure constants $C^{mn}_{ab,bc} = -\delta_{ma}\delta_{nc}$ which are ω-independent, and non-zero for all algebras in the CK family. Let us start with the case of an even $N = 2l$ and the $\frac{1}{2}N(N+1) \times \frac{1}{2}N(N+1)$ matrix M_g (3.8). We consider the minor obtained by eliminating the rows and columns associated to the following $l = N/2$ variables α_{ab}:

$$\alpha_{0N}\alpha_{1N-1}\alpha_{2N-2}\ldots\alpha_{l-2l+2}\alpha_{l-1l+1} \tag{3.27}$$

this is, for α_{0N} we discard the row and column with elements $(M_g)_{0N,kl}$ and $(M_g)_{kl,0N}$ $(\forall kl)$, etc. The dimension of this submatrix is $N^2/2$. It can be checked that in each row and in each column there is always a *single* α of the sequence (3.27) without any factor ω. Then by permuting rows and columns we can arrange the minor in order to get all those α's in the main diagonal, so its determinant is (up to a sign), a monomial:

$$\alpha_{0N}^{2(N-1)}\alpha_{1N-1}^{2(N-3)}\alpha_{2N-2}^{2(N-5)}\cdots\alpha_{l-2l+2}^{2\cdot3}\alpha_{l-1l+1}^{2\cdot1}. \tag{3.28}$$

Since this term is ω-independent, this minor is non-zero for any CK algebra. A similar procedure is applied for an odd $N = 2l-1$. Now we take out the rows and columns linked to the $l = (N+1)/2$ variables:

$$\alpha_{0N}\alpha_{1N-1}\alpha_{2N-2}\ldots\alpha_{l-2l+1}\alpha_{l-1l} \tag{3.29}$$

obtaining in this way a minor of dimension $(N^2-1)/2$. By ordering rows and columns, we get all variables appearing in the sequence (3.29) placed in the main diagonal; the determinant is (again up to a sign) the monomial:

$$\alpha_{0N}^{2(N-1)}\alpha_{1N-1}^{2(N-3)}\alpha_{2N-2}^{2(N-5)}\cdots\alpha_{l-3l+2}^{2\cdot4}\alpha_{l-2l+1}^{2\cdot2} \tag{3.30}$$

and the result follows. □

Hence, as $\dim(g) = \frac{1}{2}N(N+1)$, the upper bound for the number of algebraically independent Casimirs in any CK algebra turns out to be $[\frac{N+1}{2}]$ which coincides with the upper bound for the simple algebras where all $\omega_a \neq 0$. Since we have just obtained that number of Casimirs, we conclude that the equality in formula (3.7) holds for *all* CK algebras and theorem 2 gives *all* the Casimirs for the CK family. We remark that the flag algebra $\mathrm{so}_{0,\ldots,0}(N+1)$ is on the borderline for this behaviour: if contractions are carried out beyond this algebra, the rank of the matrix in proposition 3 might not be given by the same values.

This can be easily seen: for the Abelian algebra in $\frac{1}{2}(N+1)N$ dimensions, which can of course be reached by contracting so$(N+1)$, all generators are central elements.

If we define now the *rank of a Lie algebra* in the CK family as the number of algebraically independent Casimir invariants associated to it, then theorem 2 shows that all the CK algebras $\mathrm{SO}_{\omega_1,\ldots,\omega_N}(N+1)$ have the same rank: $N/2$ if N is even and $(N+1)/2$ if N is odd.

We recall that the first Casimir (3.22) for $s=1$ is the quadratic invariant related to the Killing–Cartan form in the case of a simple algebra. If a 'Killing–Cartan' form is defined for all CK algebras as usual:

$$\beta_{ab,cd} \equiv \beta(\Omega_{ab}, \Omega_{cd}) = \text{Trace } (\text{ad}\,\Omega_{ab} \cdot \text{ad}\,\Omega_{cd}) = \sum_{m,n,p,q=0}^{N} C^{pq}_{ab,mn} C^{mn}_{cd,pq} \tag{3.31}$$

we find, by using the structure constants (3.26), that in the basis Ω_{ab} this 'Killing–Cartan' form of the CK algebra $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ is diagonal, and its non-zero components are

$$\beta_{ab,ab} = -2(N-1)\omega_{ab} \qquad a, b = 0, \ldots, N \qquad a < b. \tag{3.32}$$

When all the $\omega_a \neq 0$ the Killing–Cartan form is regular (the algebra is simple or semisimple in the exceptional case D_2), so we can write:

$$\mathcal{C}_1 = \sum_{a,b=0}^{N} \omega_{0a}\omega_{bN}\Omega^2_{ab} = \sum_{a,b=0}^{N} \frac{\omega_{0N}}{\omega_{ab}} \Omega^2_{ab} = -2(N-1)\omega_{0N} \sum_{a,b;c,d=0}^{N} \beta^{ab,cd}\Omega_{ab}\Omega_{cd}. \tag{3.33}$$

This is the known relation giving the quadratic Casimir as the dual of the Killing–Cartan form, which holds for the case of non-zero ω_a; otherwise the Killing–Cartan form is degenerate, and the last term in (3.33) is indeterminate. However, the structure of this equation shows that in the limit of some $\omega_a \to 0$, while the Killing–Cartan form (3.32) degenerates, the Casimir $\mathcal{C}_1$ remains well defined, because the impossibility of inverting the matrix $\beta_{ab,cd}$ conspires with the factor ω_{0N} to produce a well defined limit for $\mathcal{C}_1$. Similarly, higher-order Casimir invariants are dual to the polarized form of a symmetric multilinear form.

The algebraic structure behind the W's, which allows simple expressions for the higher-order Casimirs, should be worth studying. For instance, let us consider the commutation relations among a generator Ω_{ab} and a symbol $W_{a_1a_2\ldots a_s b_1 b_2 \ldots b_s}$. There are two possibilities.

(1) If both indices a and b, or none, appear in the sequence $\{a_1a_2\ldots a_s b_1 b_2 \ldots b_s\}$, then the Lie bracket $[\Omega_{ab}, W_{a_1a_2\ldots a_s b_1 b_2 \ldots b_s}]$ is zero.

(2) If only *one* index a or b belongs to $\{a_1a_2\ldots a_s b_1 b_2 \ldots b_s\}$, then we have:

$$[\Omega_{ab}, W_{a_1a_2\ldots a_s b_1 b_2 \ldots b_s}] = (-1)^{p+1} \sqrt{\omega_{ab} \frac{\omega_{a_1b_s}\omega_{a_2b_{s-1}} \ldots \omega_{a_sb_1}}{\omega_{a'_1b'_s}\omega_{a'_2b'_{s-1}} \ldots \omega_{a'_sb'_1}}}\, W_{a'_1a'_2\ldots a'_s b'_1 b'_2 \ldots b'_s} \tag{3.34}$$

where the new set of indices $\{a'_1a'_2\ldots a'_s b'_1 b'_2 \ldots b'_s\}$ is obtained by first writing the sequence $\{a, b; a_1a_2\ldots a_s b_1 b_2 \ldots b_s\}$ in increasing order, and then dropping the common index. In the factor $(-1)^{p+1}$, p means the minimum number of transpositions needed to bring the sequence $\{ab; a_1a_2\ldots a_s b_1 b_2 \ldots b_s\}$ into increasing order; for instance, in the sequence $\{13; 0234\} \equiv \{0124\}$ $p=3$, while in $\{15; 1234\} \equiv \{2345\}$ $p=4$. Notice that all $\omega_{a'b'}$ in the denominator cancel, and leave under the square root a perfect square product of ω's.

Repeated use of this procedure would give the commutation relations between two W-symbols. We do not write here the general expressions, but some examples are given in the next section.

4. Examples

4.1. Results for $N = 2,3,4,5$

In the sequel we elaborate upon the results of the above section by writing explicitly the Casimir invariants of $\mathrm{so}_{\omega_1,\ldots,\omega_N}(N+1)$ up to $N = 5$. These expressions should be compared with those obtained in any approach giving the invariants directly in terms of the generators Ω_{ab} (without using the W-symbols) which are cumbersome as soon as N grows.

4.1.1. $N = 2$. There is only one invariant:

$$\mathcal{C}_1 = \omega_2\Omega_{01}^2 + \Omega_{02}^2 + \omega_1\Omega_{12}^2. \tag{4.1}$$

4.1.2. $N = 3$. There are two invariants and the first relevant W-symbol (3.20) appears:

$$\begin{aligned}\mathcal{C}_1 = \omega_2\omega_3\Omega_{01}^2 &+\omega_3\Omega_{02}^2 &&+\Omega_{03}^2\\ &+\omega_1\omega_3\Omega_{12}^2 &&+\omega_1\Omega_{13}^2\\ & &&+\omega_1\omega_2\Omega_{23}^2\end{aligned} \tag{4.2}$$

$$\begin{aligned}\mathcal{C} \equiv W_{0123} &= \omega_{12}\Omega_{23}W_{01} - \Omega_{13}W_{02} + \Omega_{03}W_{12}\\ &= \omega_2\Omega_{23}\Omega_{01} - \Omega_{13}\Omega_{02} + \Omega_{03}\Omega_{12}.\end{aligned} \tag{4.3}$$

4.1.3. $N = 4$. From a physical point of view this is a rather interesting case since the CK family $\mathrm{so}_{\omega_1,\ldots,\omega_4}(5)$ contains the (3+1)-kinematical algebras. There are two invariants:

$$\begin{aligned}\mathcal{C}_1 = \omega_2\omega_3\omega_4\Omega_{01}^2 &+\omega_3\omega_4\Omega_{02}^2 &&+\omega_4\Omega_{03}^2 &&+\Omega_{04}^2\\ &+\omega_1\omega_3\omega_4\Omega_{12}^2 &&+\omega_1\omega_4\Omega_{13}^2 &&+\omega_1\Omega_{14}^2\\ & &&+\omega_1\omega_2\omega_4\Omega_{23}^2 &&+\omega_1\omega_2\Omega_{24}^2\\ & && &&+\omega_1\omega_2\omega_3\Omega_{34}^2\end{aligned} \tag{4.4}$$

$$\mathcal{C}_2 = \omega_{24}W_{0123}^2 + \omega_{23}W_{0124}^2 + W_{0134}^2 + \omega_{12}W_{0234}^2 + \omega_{02}W_{1234}^2 \tag{4.5}$$

where from (3.20) we have

$$\begin{aligned}W_{0123} &= \omega_{12}\Omega_{23}\Omega_{01} - \Omega_{13}\Omega_{02} + \Omega_{03}\Omega_{12}\\ W_{0124} &= \omega_{12}\Omega_{24}\Omega_{01} - \Omega_{14}\Omega_{02} + \Omega_{04}\Omega_{12}\\ W_{0134} &= \omega_{13}\Omega_{34}\Omega_{01} - \Omega_{14}\Omega_{03} + \Omega_{04}\Omega_{13}\\ W_{0234} &= \omega_{23}\Omega_{34}\Omega_{02} - \Omega_{24}\Omega_{03} + \Omega_{04}\Omega_{23}\\ W_{1234} &= \omega_{23}\Omega_{34}\Omega_{12} - \Omega_{24}\Omega_{13} + \Omega_{14}\Omega_{23}.\end{aligned} \tag{4.6}$$

As an application of (3.34) we write the non-zero commutation relations among the 10 generators of $\mathrm{so}_{\omega_1,\omega_2,\omega_3,\omega_4}(5)$ and the five W-symbols (4.6):

$$\begin{array}{lll}
[\Omega_{04}, W_{0123}] = -\omega_{02}W_{1234} & [\Omega_{03}, W_{0124}] = \omega_{02}W_{1234} & [\Omega_{02}, W_{0134}] = -\omega_{02}W_{1234}\\
[\Omega_{14}, W_{0123}] = \omega_{12}W_{0234} & [\Omega_{13}, W_{0124}] = -\omega_{12}W_{0234} & [\Omega_{12}, W_{0134}] = \omega_{12}W_{0234}\\
[\Omega_{24}, W_{0123}] = -W_{0134} & [\Omega_{23}, W_{0124}] = W_{0134} & [\Omega_{23}, W_{0134}] = -\omega_{23}W_{0124}\\
[\Omega_{34}, W_{0123}] = W_{0124} & [\Omega_{34}, W_{0124}] = -\omega_{34}W_{0123} & [\Omega_{24}, W_{0134}] = \omega_{24}W_{0123}\\
[\Omega_{01}, W_{0234}] = \omega_{01}W_{1234} & [\Omega_{01}, W_{1234}] = -W_{0234} & [\Omega_{12}, W_{0234}] = -W_{0134}\\
[\Omega_{02}, W_{1234}] = W_{0134} & [\Omega_{13}, W_{0234}] = \omega_{23}W_{0124} & [\Omega_{03}, W_{1234}] = -\omega_{23}W_{0124}\\
[\Omega_{14}, W_{0234}] = -\omega_{24}W_{0123} & [\Omega_{04}, W_{1234}] = \omega_{24}W_{0123}. &
\end{array} \tag{4.7}$$

As these expressions show, the W-symbols can be thought of as a kind of 'Pauli–Lubanski' components for Lie algebras in the CK family. To see this, notice that when the constants $(\omega_1, \omega_2, \omega_3, \omega_4) \equiv (0, -1, +1, +1)$, the CK Lie algebra $\mathrm{so}_{0,-1,+1,+1}(5)$ is isomorphic to the Poincaré algebra $\mathrm{iso}(3, 1)$. In this case there are *five* W-symbols; due to the vanishing of the constant ω_1, W_{1234} is missing in the expression of the fourth-order Casimir (4.5), and the remaining *four* are the components of the standard Pauli–Lubanski operator (see (4.18) below).

From (4.7) it is straightforward to find the Lie brackets of the W-symbols among themselves:

$$\begin{aligned}
[W_{0123}, W_{0124}] &= \omega_{12}\Omega_{01}W_{0134} + \omega_{12}\Omega_{02}W_{0234} + \omega_{02}\Omega_{12}W_{1234}\\
[W_{0123}, W_{0134}] &= -\omega_{13}\Omega_{01}W_{0124} + \omega_{12}\Omega_{03}W_{0234} + \omega_{02}\Omega_{13}W_{1234}\\
[W_{0123}, W_{0234}] &= -\omega_{23}\Omega_{02}W_{0124} - \Omega_{03}W_{0134} + \omega_{02}\Omega_{23}W_{1234}\\
[W_{0123}, W_{1234}] &= -\omega_{23}\Omega_{12}W_{0124} - \Omega_{13}W_{0134} - \omega_{12}\Omega_{23}W_{0234}\\
[W_{0124}, W_{0134}] &= \omega_{14}\Omega_{01}W_{0123} + \omega_{02}\Omega_{14}W_{1234} + \omega_{12}\Omega_{04}W_{0234}\\
[W_{0124}, W_{0234}] &= \omega_{24}\Omega_{02}W_{0123} - \Omega_{04}W_{0134} + \omega_{02}\Omega_{24}W_{1234}\\
[W_{0124}, W_{1234}] &= \omega_{24}\Omega_{12}W_{0123} - \Omega_{14}W_{0134} - \omega_{12}\Omega_{24}W_{0234}\\
[W_{0134}, W_{0234}] &= \omega_{24}\Omega_{03}W_{0123} + \omega_{23}\Omega_{04}W_{0124} + \omega_{03}\Omega_{34}W_{1234}\\
[W_{0134}, W_{1234}] &= \omega_{24}\Omega_{13}W_{0123} + \omega_{23}\Omega_{14}W_{0124} - \omega_{13}\Omega_{34}W_{0234}\\
[W_{0234}, W_{1234}] &= \omega_{24}\Omega_{23}W_{0123} + \omega_{23}\Omega_{24}W_{0124} + \omega_{23}\Omega_{34}W_{0134}.
\end{aligned} \tag{4.8}$$

4.1.4. $N = 5$. The three invariants are given by:

$$\begin{aligned}
\mathcal{C}_1 = \omega_{15}\Omega_{01}^2 \quad &+\omega_{25}\Omega_{02}^2 \quad +\omega_{35}\Omega_{03}^2 \quad +\omega_{45}\Omega_{04}^2 \quad +\Omega_{05}^2\\
&+\omega_{01}\omega_{25}\Omega_{12}^2 \quad +\omega_{01}\omega_{35}\Omega_{13}^2 \quad +\omega_{01}\omega_{45}\Omega_{14}^2 \quad +\omega_{01}\Omega_{15}^2\\
&\qquad +\omega_{02}\omega_{35}\Omega_{23}^2 \quad +\omega_{02}\omega_{45}\Omega_{24}^2 \quad +\omega_{02}\Omega_{25}^2\\
&\qquad\qquad +\omega_{03}\omega_{45}\Omega_{34}^2 \quad +\omega_{03}\Omega_{35}^2\\
&\qquad\qquad\qquad +\omega_{04}\Omega_{45}^2
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\mathcal{C}_2 = \omega_{35}\omega_{24}W_{0123}^2 &+ \omega_{25}W_{0124}^2 + \omega_{24}W_{0125}^2 + \omega_{35}W_{0134}^2 + \omega_{34}W_{0135}^2 + W_{0145}^2\\
&+\omega_{12}\omega_{35}W_{0234}^2 + \omega_{12}\omega_{34}W_{0235}^2 + \omega_{12}W_{0245}^2 + \omega_{13}W_{0345}^2 + \omega_{02}\omega_{35}W_{1234}^2\\
&+\omega_{02}\omega_{34}W_{1235}^2 + \omega_{02}W_{1245}^2 + \omega_{03}W_{1345}^2 + \omega_{02}\omega_{13}W_{2345}^2
\end{aligned} \tag{4.10}$$

$$\mathcal{C} \equiv W_{012345} = \omega_{24}\Omega_{45}W_{0123} - \omega_{23}\Omega_{35}W_{0124} + \Omega_{25}W_{0134} - \Omega_{15}W_{0234} + \Omega_{05}W_{1234}. \tag{4.11}$$

In the above expressions it can be noticed how all the contractions $\omega_a \to 0$ are always well defined and lead to non-trivial results. For the most contracted algebra $\mathrm{so}_{0,0,0,0,0}(6)$, for instance, we get:

$$\mathcal{C}_1 = \Omega_{05}^2 \qquad \mathcal{C}_2 = W_{0145}^2 \qquad \mathcal{C} = W_{012345}. \tag{4.12}$$

And for N arbitrary, the Casimirs for this most contracted CK algebra are:

$$\mathcal{C}_1 = \Omega_{0N}^2 \equiv W_{0N}^2 \qquad \mathcal{C}_2 = W_{01(N-1)N}^2 \qquad \mathcal{C}_3 = W_{012(N-2)(N-1)N}^2, \ldots. \tag{4.13}$$

In fact, even if the appearance of the factors ω_{ab} seems rather haphazard when written in specific cases, such as in (4.10), they are indeed easily reconstructed from scratch, without reference to the general expressions (3.24). If here $[X]$ denotes the dimension of X, dimensional homogeneity of commutation relations requires that $[\Omega_{ab}] = [\omega_{ab}]^{1/2}$, so $[W_{ab}^2] = [\omega_{ab}]$, $[W_{a_1a_2b_1b_2}^2] = [\omega_{a_1b_2}][\omega_{a_2b_1}]$, etc. By simply recalling the W term (4.13) entering into any Casimir *without* any ω_{ab} factor, then the coefficient of any other W^2 is unambiguously derived by simply requiring dimensional homogeneity for all W^2 terms in the Casimirs and recalling that all terms enter with the same global sign there. Signs coming from signatures which appear in the standard known cases (in the Minkowski square of the Pauli–Lubanski vector, for instance), are hidden inside the ω_{ab} themselves.

4.2. *Casimirs for (3+1)-kinematical algebras*

After this algebraic description of the structure of the invariants of the CK algebras we focus on the most important kinematical algebras included in the $\mathrm{so}_{\omega_1,\omega_2,\omega_3,\omega_4}(5)$ family. Let H, P_i, K_i and J_i ($i = 1, 2, 3$) the usual generators of time translation, space translations, boosts and spatial rotations, respectively. Under the following identification with the 'abstract' generators Ω_{ab}:

$$\begin{array}{llll} H = \Omega_{01} & P_i = \Omega_{0i+1} & K_i = \Omega_{1i+1} & i = 1, 2, 3 \\ J_1 = \Omega_{34} & J_2 = -\Omega_{24} & J_3 = \Omega_{23} & \end{array} \tag{4.14}$$

we can interpret the six CK algebras $\mathrm{so}_{\omega_1,\omega_2,+,+}(5)$ with $\omega_2 \leqslant 0$ as the Lie algebras of the groups of motions of different (3+1) spacetime models [16]. Note that we have fixed the two coefficients ω_3 and ω_4 to $+1$ (this is a consequence of the space isotropy). The two remaining ones have a definite physical interpretation: ω_1 is the constant curvature of the spacetime which appears here as the homogeneous space given in (2.10), which is the quotient $\mathcal{X}_0 = \mathrm{SO}_{\omega_1,\omega_2,+,+}(5)/\mathrm{SO}_{\omega_2,+,+}(4)$, where $\mathrm{SO}_{\omega_2,+,+}(4)$ is the subgroup generated by the subalgebra $h_0 = \langle K_i, J_i\rangle$: this is the subalgebra of isotopy of a point in spacetime. Similarly, ω_2 is the curvature of the space of time-like lines in spacetime, $\mathcal{X}_{01} = \mathrm{SO}_{\omega_1,\omega_2,+,+}(5)/(\mathrm{SO}_{\omega_1}(2)\otimes\mathrm{SO}_{+,+}(3))$, where now $\mathrm{SO}_{\omega_1}(2)\otimes\mathrm{SO}_{+,+}(3)$ is the isotopy subgroup of a time-like line, generated by $h_{01} = \langle H, J_i\rangle$. This curvature is linked to the fundamental constant c of relativistic theories as $\omega_2 = -1/c^2$. To make the comparison easier for these cases, we shall write the two constants involved in the 'kinematical' subfamily of CK algebras as: $\kappa \equiv \omega_1$, $-1/c^2 \equiv \omega_2$.

In this notation the commutation rules (2.2) of $\mathrm{so}_{\kappa,-1/c^2,+,+}(5)$ now read:

$$\begin{array}{lll} [J_i, J_j] = \varepsilon_{ijk}J_k & [J_i, P_j] = \varepsilon_{ijk}P_k & [J_i, K_j] = \varepsilon_{ijk}K_k \\ [P_i, P_j] = -\dfrac{\kappa}{c^2}\varepsilon_{ijk}J_k & [K_i, K_j] = -\dfrac{1}{c^2}\varepsilon_{ijk}J_k & [P_i, K_j] = -\dfrac{1}{c^2}\delta_{ij}H \\ [H, P_i] = \kappa K_i & [H, K_i] = -P_i & [H, J_i] = 0 \qquad i, j, k = 1, 2, 3. \end{array} \tag{4.15}$$

The limit $\kappa \to 0$ (space-time contraction) gives rise to the *flat* universes (Minkowski and Galilei) coming from the curved ones (de Sitter and Newton–Hooke); in terms of the 'universe radius' $R := \frac{1}{\sqrt{\kappa}}$ or $R := \frac{1}{\sqrt{-\kappa}}$, this is usually made as $R \to \infty$. The limit $c \to \infty$ (speed-space contraction) leads to 'absolute-time' spacetimes (Newton–Hooke and Galilei) coming from 'relative-time' ones (de Sitter and Minkowski).

In this context, the Casimir invariants (4.4) and (4.5) adopt the form

$$\mathcal{C}_1 = P_1^2 + P_2^2 + P_3^2 - \frac{1}{c^2}H^2 + \kappa(K_1^2 + K_2^2 + K_3^2) - \frac{\kappa}{c^2}(J_1^2 + J_2^2 + J_3^2) \tag{4.16}$$

$$\mathcal{C}_2 = W_{0123}^2 + W_{0124}^2 + W_{0134}^2 - \frac{1}{c^2}W_{0234}^2 - \frac{\kappa}{c^2}W_{1234}^2 \tag{4.17}$$

Table 1. The Casimir invariants of $\mathrm{so}_{\kappa,-1/c^2,+,+}(5)$.

Oscillating Newton–Hooke $(+,0,+,+)\kappa=1, c=\infty$ $t_6(\mathrm{so}(3)\oplus\mathrm{so}(2))$	Galilei $(0,0,+,+)\kappa=0, c=\infty$ $\mathrm{iiso}(3)$	Expanding Newton–Hooke $(-,0,+,+)\kappa=-1, c=\infty$ $t_6(\mathrm{so}(3)\oplus\mathrm{so}(1,1))$
$\mathcal{C}_1=P_1^2+P_2^2+P_3^2$ $\quad+K_1^2+K_2^2+K_3^2$ $\mathcal{C}_2=W_{0123}^2+W_{0124}^2+W_{0134}^2$	$\mathcal{C}_1=P_1^2+P_2^2+P_3^2$ $\mathcal{C}_2=W_{0123}^2+W_{0124}^2+W_{0134}^2$	$\mathcal{C}_1=P_1^2+P_2^2+P_3^2$ $\quad-K_1^2-K_2^2-K_3^2$ $\mathcal{C}_2=W_{0123}^2+W_{0124}^2+W_{0134}^2$
Anti-de Sitter $\mathrm{so}(3,2)$ $(+,-,+,+)\kappa=1, c=1$	Poincaré $\mathrm{iso}(3,1)$ $(0,-,+,+)\kappa=0, c=1$	de Sitter $\mathrm{so}(4,1)$ $(-,-,+,+)\kappa=-1, c=1$
$\mathcal{C}_1=P_1^2+P_2^2+P_3^2-H^2$ $\quad+K_1^2+K_2^2+K_3^2$ $\quad-J_1^2-J_2^2-J_3^2$ $\mathcal{C}_2=W_{0123}^2+W_{0124}^2+W_{0134}^2$ $\quad-W_{0234}^2-W_{1234}^2$	$\mathcal{C}_1=P_1^2+P_2^2+P_3^2-H^2$ $\mathcal{C}_2=W_{0123}^2+W_{0124}^2+W_{0134}^2$ $\quad-W_{0234}^2$	$\mathcal{C}_1=P_1^2+P_2^2+P_3^2-H^2$ $\quad-K_1^2-K_2^2-K_3^2$ $\quad+J_1^2+J_2^2+J_3^2$ $\mathcal{C}_2=W_{0123}^2+W_{0124}^2+W_{0134}^2$ $\quad-W_{0234}^2+W_{1234}^2$

where

$$\begin{aligned}
W_{0123} &= -\frac{1}{c^2}HJ_3 - P_1K_2 + P_2K_1\\
W_{0124} &= \frac{1}{c^2}HJ_2 - P_1K_3 + P_3K_1\\
W_{0134} &= -\frac{1}{c^2}HJ_1 - P_2K_3 + P_3K_2\\
W_{0234} &= P_1J_1 + P_2J_2 + P_3J_3\\
W_{1234} &= K_1J_1 + K_2J_2 + K_3J_3.
\end{aligned} \tag{4.18}$$

In table 1 we display these six kinematical algebras together with their invariants according to the values of $(\kappa, \frac{-1}{c^2}, +, +)$. The limit transitions among them can be clearly appreciated; note that some W-symbols (4.18) are 'internally contracted' in the case of $c=\infty$.

5. Concluding remarks

The Casimir invariants play a prominent role in any problem where a Lie algebra and its enveloping algebra appear. One example of current active interest is the theory of quantum groups; explicit deformations of the W-symbols can be found in the deformed commutation relations of the quantum CK algebras $U_z\mathrm{so}_{0,\omega_2,\ldots,\omega_N}(N+1)$ with $\omega_1=0$, and indeed the study of Casimirs in the classical undeformed algebras we have presented here underlies the expressions for the deformed Casimirs in [24].

A more classical application concerns the expansions processes which can be seen as the opposite situation of a contraction limit [25]. While the contractions make some structure constants of a Lie algebra vanish which therefore gets more Abelian, the expansions start from a Lie algebra, with some Lie brackets typically equal to zero, and end up with another less Abelian algebra, which is usually realized as a Lie subalgebra of the universal enveloping algebra of the initial Lie algebra. The more known transitions of this kind are the rank-one expansions which allow us to obtain the simple $\mathrm{so}(p,q)$ algebras starting from inhomogeneous $\mathrm{iso}(p,q)$; the name rank-one refers to the rank of the homogeneous spaces behind these expansions (the spaces with a metric of signature (p,q) and constant curvature, the expansion going from flat to non-flat spaces) [25, 26]. In our framework this fact is equivalent to 'creating' a non-zero coefficient ω_1 out of the case $\omega_1 = 0$. Technically, these expansions only involve the quadratic Casimir $\mathcal{C}_1$ to get the correct Lie subalgebra in the enveloping algebra to be expanded (within an irreducible representation). An extension of these results for high-order expansions would be of great interest. The rank-two expansions would go from $t_r(\mathrm{so}(p,q) \oplus \mathrm{so}(p',q'))$ (Newton–Hooke algebras) to $\mathrm{so}(p,q)$ algebras, that is, they would introduce curvature into the space of lines, in the same way as the rank-one expansions go from flat to curved spaces of points. Here the two first Casimir invariants $\mathcal{C}_1$ and $\mathcal{C}_2$ should participate, and it is reasonable to guess that the explicit introduction of the constants ω_a may help in the choice of the correct expansion procedure which, as far as we know, is still unknown for higher rank spaces.

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